LESSON 37

INVERSE OF A MATRIX

In this lesson, we will learn finding out the inverse of a matrix. When solving real number equations like *ax=b*, we divide both sides by *a* and get $x = \frac{b}{a}$, but when dealing with matrices, if we have AX = B, where A and B are matrices, we can't divide both sides by B. Because, in matrices, division is not defined. So we use inverse of a matrix to find out the value of X, what we will do is to multiply both sides by A^{-1},

$(A^{-1}).(AX) = (A^{-1}).B$

As we know that in real numbers when a number is multiplied by its multiplicative inverse, it results into multiplicative identity that is 1. Same is the rule in matrices. When a matrix is multiplied by its inverse it results into the identity matrix I. So $A . A^{-1} = I$

It means the equation

$(A^{-1}).(AX) = (A^{-1}).B$

Can be written as,

$I .X = (A^{-1}).B$

$X = A^{-1} B$

So how can we find the inverse of a matrix. That's what we are going to study now. We can not find inverse of every matrix. There are some **conditions of finding inverse**.

- The first condition is that the matrix should a square matrix.
- The second condition is that determinant of the matrix should not be equal to zero.

Any square matrix whose inverse exists is called a **non-singular matrix or invertible matrix**, where as a matrix that does not have an inverse is called a **singular matrix.**

Let's start by finding the inverse of a 2x2 matrix.

Its very easy to find inverse of a 2x2 matrix. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then inverse of A can be find out by the formula,

$A^{-1} = \frac{1}{|A|} . \text{Adj } A$

Where Adj A = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and |A| = ad – bc

Let's try out an example.

Example 1:

Find inverse of A, If A = $\begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix}$.

Solution:

The very first step in finding an inverse is to calculate the determinant of the matrix and make sure that its not equal to zero. So let's find the determinant.

$|A| = \begin{vmatrix} 5 & 2 \\ -3 & 1 \end{vmatrix}$

|A| = (5)(1) – (-3)(2)

$|A| = 5 + 6 = 11 \neq 0$

$|A| \neq 0$ indicates that inverse of A exists.

Now lets find out Adj A,

Adj A = $\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$

Now we will put the values of |A| and Adj A in the formula of inverse.

$A^{-1} = \frac{1}{|A|} . \text{Adj } A$

$A^{-1} = \frac{1}{11} * \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{1}{11} & \frac{-2}{11} \\ \frac{3}{11} & \frac{5}{11} \end{bmatrix}$

This is the way, we find inverse of a 2x2 matrix. Now we will learn to find the inverse of matrices of order 3 and higher.

We will use the same formula for inverse that is,

$A^{-1} = \frac{1}{|A|} . \text{Adj } A$

But we will use Cofactors to find out the determinant of A and Adjoint of A for 3x3 matrices.

We have already discussed the method of finding determinant of 3x3 matrix with the help of cofactors. The formula we use is

$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

And Adjoint of A is calculated by finding the transpose of the matrix of cofactors.

Matrix of cofactors is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Adjoint of A will be the transpose of this matrix.

$$\text{Adj A} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Let's solve an example to understand it.

Example 2:

Find A^{-1} if,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Solution:

We will first find the cofactors of the elements of the matrix A.

$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}$

$A_{11} = (-1)^2 [(2)(1) – (-1)(1)]$

$A_{11} = (1) (2 + 1)$

$A_{11} = 3$

$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$

$A_{12} = (-1)^3 [(0)(1) – (1)(1)]$

$A_{12} = (-1)\ (0 - 1)$

$A_{12} = 1$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix}$

$A_{13} = (-1)^{4}\ [\ (0)(-1) - (1)(2)\ ]$

$A_{13} = (1)\ (\ 0 - 2\ )$

$A_{13} = -2$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}$

$A_{21} = (-1)^{3}\ [\ (0)(1) - (-1)(2)\ ]$

$A_{21} = (-1)\ [\ 0 - (-2)\ ]$

$A_{21} = (-1)\ [\ 0 + 2\ ]$

$A_{21} = -2$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$

$A_{22} = (-1)^{4}\ [\ (1)(1) - (1)(2)\ ]$

$A_{22} = (1)\ [\ 1 - 2\ ]$

$A_{22} = (1)\ [\ -1\ ]$

$A_{22} = -1$

$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}$

$A_{23} = (-1)^{5}\ [\ (1)(-1) - (1)(0)\ ]$

$A_{23} = (-1)\ [\ -1 - 0\ ]$

$A_{23} = (-1)\ (-1)$

$A_{23} = 1$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix}$

$A_{31} = (-1)^4 [(0)(1) - (2)(2)]$

$A_{31} = (1) [0 - (4)]$

$A_{31} = (1) (-4)$

$A_{31} = -4$

$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

$A_{32} = (-1)^5 [(1)(1) - (0)(2)]$

$A_{32} = (-1) [1 - 0]$

$A_{32} = (-1) (1)$

$A_{32} = -1$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$

$A_{33} = (-1)^6 [(1)(2) - (0)(0)]$

$A_{33} = (1) [2 - 0]$

$A_{33} = (1) (2)$

$A_{33} = 2$

So matrix of cofactors will be

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$$

Adj A will be transpose of the matrix of cofactors.

$$\text{Adj A} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

Now we will find the determinant of A using the cofactors with the help of the formula,

$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

|A| = 1 (3) + 0 (1) + 2 (-2)

|A| = 3 + 0 – 4

|A| = -1

Now we will find the inverse of A by the formula by putting values of Adj A and |A|,

$A^{-1} = \frac{1}{|A|}$. Adj A

$$A^{-1} = \frac{1}{(-1)} . \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$A^{-1} = (-1) . \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$$

It shows that Inverse of a matrix is the product of adjoint of matrix and reciprocal of the determinant. Where as Adjoint of a matrix is the transpose of the matrix of cofactors. Remember whenever a matrix is multiplied by its inverse, it results into identity matrix.